operation has proven to be extremely helpful in plotting stability boundaries for systems in which the elements of P are functions of parameters that are being varied. The scheme has a tendency to break down for large matrices (25×25 or larger) due to the extremely large numbers generated in taking the determinant Δ_{N-1}. This can be partially alleviated by using double precision and by normalizing the original matrix P so that the numerical size of the elements c_i is decreased. The particular way in which P is normalized is not crucial so long as the eigenvalues are not affected except within a multiplicative constant.

Numerical Example

For an example of a dynamic system as described in the preceding, we assume that the equation of motion is of the following form.

$$I\ddot{x} + (\delta S + \epsilon I)\dot{x} + (K + aS)x = 0$$

where I is the $n \times n$ identity matrix and elements S and K are given

$$S_{i,j} = \begin{cases} 0, & i=j \\ +1, & i<j \\ -1, & i>j \end{cases} \Bigg\} i,j = 1,2,\ldots,n$$

$$K_{i,j} = \begin{cases} 0, & i \neq j \\ k^{2i-2}, & i=j \end{cases}$$

and where δ, ϵ, k and a are scalars reflecting the strength of each matrix. The stability of motions described by x is reflected by the eigenvalues of the $N \times N$ matrix P where

$$P = \begin{bmatrix} 0 & I \\ -(K+aS) & -(\delta S + \epsilon I) \end{bmatrix}$$

and $N = 2n$. In Fig. 1, we compare the relative CPU time for extracting the eigenvalues using the QR algirithm and for calculating both c_i for $i = 1,2,\ldots,N+1$ and Δ_{N-1}. As is evident, the calculation of Δ_{N-1} is from 15 to 30% faster for $N > 3$. The percentage improvement tends to decrease as N increases, but the actual difference between the CPU times (not shown) increases monotonically with N. In Fig. 2, we show $Re(\lambda_1)$, the real part of the eigenvalue associated with the lowest frequency and Δ_5 for $n = 3$, $\epsilon = 0.001$, $a = -0.5$, $k = 5$, and δ varying. The crossing of a stability boundary is evident in the change in sign of $Re(\lambda_1)$ and Δ_5. The two methods predict the same location for the stability boundary as indicated by their crossing of the horizontal axis at the same value of δ.

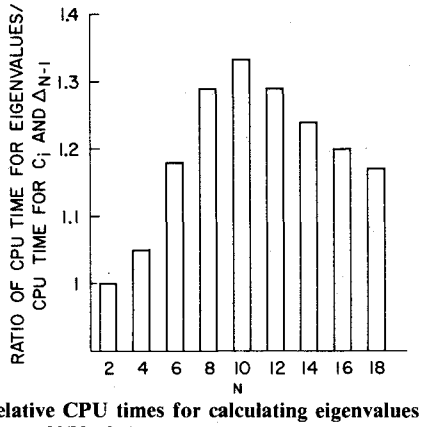

Fig. 1 Relative CPU times for calculating eigenvalues and stability information vs $N(N = 2n)$.

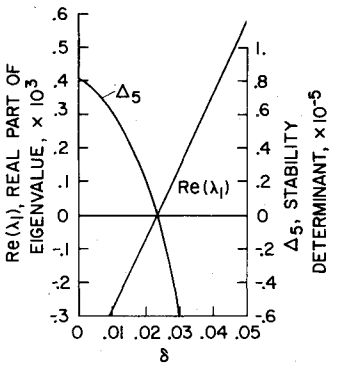

Fig. 2 $Re\ (\lambda_1)$ and Δ_5 vs δ for $a = -\frac{1}{2}$, $k = 5$, $\epsilon = 0.001$, $n = 3$.

Concluding Remarks

A simplified numerical scheme is presented which enables the analyst to determine the stability of a linear system by checking the sign of one number. The scheme has distinct advantages. Not only is it faster than extracting eigenvalues, but it also has a simpler application in determining stability. These advantages facilitate the generation of stability boundaries in parameter studies. It has a disadvantage of tending to break down for large matrices. This can be overcome partly by using double precision and by normalizing the matrix from which stability information is derived.

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Compressible Turbulent Boundary Layers with Injection

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Nomenclature

c_f = skin-friction coefficient
F = injection parameter $(\rho_w v_w / \rho_1 U_1)$ based on freestream conditions
m = injection parameter $(\rho_w v_w / \rho_o U_o)$ based on freestream conditions upstream of the pressure gradient

Introduction

As part of a general study[1-3] of compressible turbulent boundary layers with fluid injection at the surface, we

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have recently completed two new experiments investigations. One of these, by G. D. Thomas,[4] studied the effects of favorable and adverse pressure gradients. The other, by P. G. Marriott,[5] was concerned with the boundary-layer development downstream of a sharp change in injection rate. In both investigations complete velocity and temperature profiles were measured through the region of interest and some measurements of skin-friction were obtained. In conjunction with the experiments two prediction methods were used. One of these[6] used a simple eddy viscosity model and is similar to that developed by Cebeci and Smith,[7] while the other is a direct extension of the turbulence energy method developed by Bradshaw.[8] Both methods were used to predict all the measured boundary-layer developments, and the present Note presents the main points of interest from a comparison of the measured and predicted results.

It should be pointed out that in a previous paper[9] the results from the two prediction methods were compared with experimental results for flat-plate layers with various injection rates. In general it was found that there was close agreement between the results from the prediction methods and good overall agreement with measured profiles. The present paper therefore concentrates on comparisons of results in nonequilibrium conditions.

The pressure gradients were obtained by modifying a liner designed for $M=2.5$. In the case of the adverse gradient the Mach number fell linearly from $M=2.5$ at $x=50$ mm (Fig. 1) to $M=2.1$ at $x=150$ mm, while in the favorable gradient the increase in Mach number was from 2.5 to 3.0 in the same distance. In all cases the wall temperature was slightly below the stagnation temperature in the freestream and so there was a small heat transfer to the air from the wall. The boundary layer was fully turbulent at all test conditions, and in all tests R_θ was greater than 10,000.

Pressure Gradient Results

For all the flows with favorable pressure gradients the calculated profiles from both prediction methods were almost identical and were in excellent agreement with the measured profiles. However, for the adverse pressure gradient (Fig. 1) there are significant differences between the two predictions, with the results from the eddy viscosity model being slightly closer to the measured profiles with zero injection, although toward the downstream end of the development neither method is completely satisfactory. With increase in injection rate the differences between the two predictions increase as do the discrepancies with the measured profiles. In particular for $m=0.003$ the measured velocities near the wall are much higher than the predicted values. In fact it is interesting to note that although the initial profiles ($x=0$) in both calculation methods were matched to the measured profile at $x=0$, both calculated profiles revert to characteristic boundary-layer profiles at $x=25$ mm. The reasons for the unusual nature of the profiles near the wall are unknown, although similar profiles have been measured in various conditions in the current research program.[1,2]

For all the pressure gradients with, and without, injection the skin-friction coefficients as calculated by both methods agreed to better than 0.0001. The skin-friction without injection was measured by small razor blade devices[10] with an estimated accuracy of ± 0.0001, and in general the measured values agreed with the prediction to within this accuracy, although in the case of the adverse gradient the measured values were definitely higher than the predicted values. Skin-friction coefficients with injection could only be obtained from a momentum balance and are thus subject to larger errors than mentioned above. However, the levels of the measured values were in good agreement with the prediction.

Flow Downstream of a Change in Injection Rate

Figure 2 shows the measured and calculated profiles on a surface with zero injection upstream of $x=0$ and with a uniform injection rate of $F=0.0045$ downstream at a freestream Mach number of 1.8. The agreement between the measured and predicted profiles is very good, particularly near the front of the injection region where the sharp fall in velocity near the surface is predicted accurately. At $M=3.6$ (not shown) the measured and predicted profiles also show good agreement at the blowing rates up to $F=0.0029$.

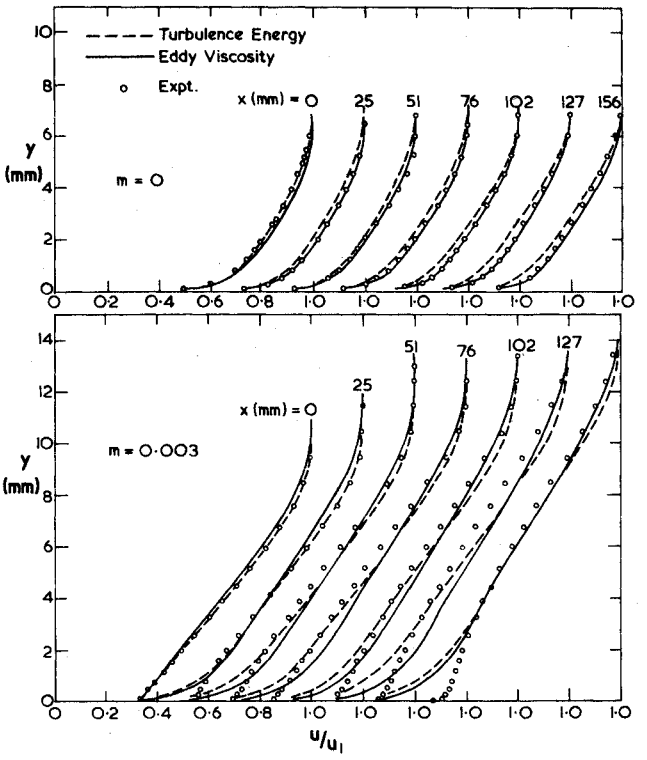

Fig. 1 Velocity profiles in an adverse pressure gradient.

Figure 2 also shows the measured and predicted profiles§ for the development of the boundary layer on a solid surface downstream of a region of injection. The initial development of the profile downstream of the injection is predicted well, but further downstream the agreement becomes worse, with the measured profile being less full than the predicted profile.

The top figure of Fig. 3 shows the calculation of skin-friction coefficients for various blowing rates at $M=1.8$, and it can be seen that the agreement between the results from the two methods are excellent. No direct measurements of skin-friction coefficient could be made in the experiments, but the lower figure shows the difference between the reading of a flattened pitot tube placed on the surface and the static pressure difference at the same point. While we would not claim that this pressure difference is directly proportional to the skin friction, it is obviously related to it. It will be noted that the general shape and relative levels of the curves in the upper and lower figures are similar. In particular the main changes in the predicted skin friction and in level of the pressure difference take place within 25 mm of the junction, i.e., within five boundary-layer thicknesses. In general the same similarity between the predicted skin friction and the pressure difference occurred at $M=3.6$.

Temperature Profiles

The energy equation is not solved in the present version of Bradshaw's turbulence energy method. Instead the tem-

§The predictions start from the measured profile 25 mm upstream of the junction. On the porous surface the skin friction was very low, and it was found that the turbulence energy method tended to predict negative skin friction at the sudden change in surface boundary condition so that the calculation could not be completed.

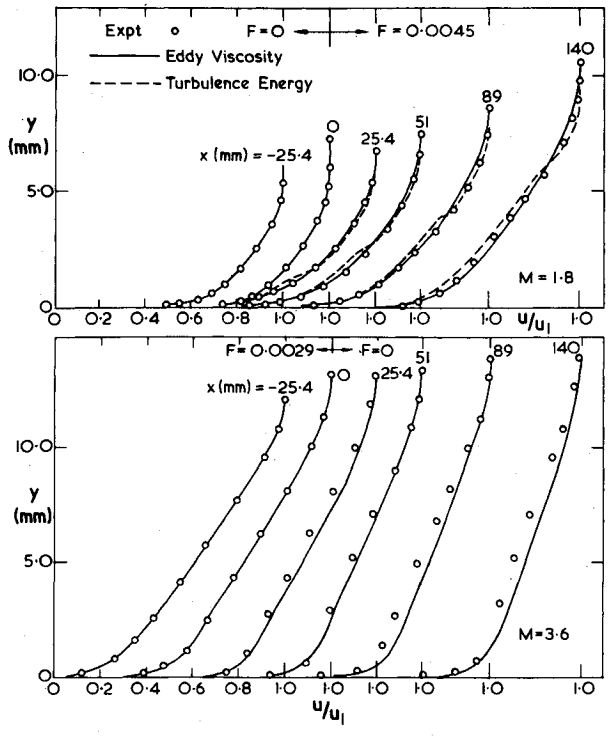

Fig. 2 Velocity profiles at a change in injection rate.

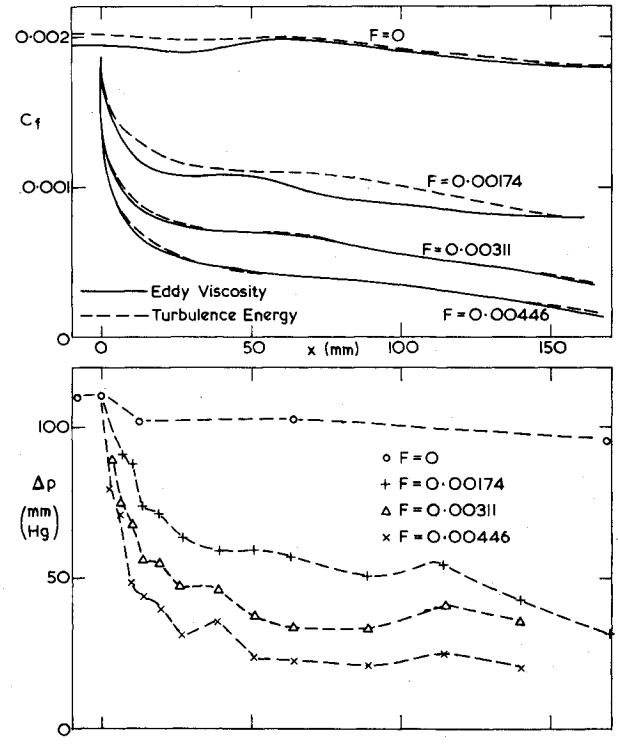

Fig. 3 Skin-friction results: solid to porous surface.

perature is assumed to be related to the velocity by the Crocco relation, i.e.

$$\frac{T}{T_1}=\frac{T_w}{T_1}+\left(\frac{T_r}{T_1}-\frac{T_w}{T_1}\right)\frac{u}{U_1}+\left(1-\frac{T_r}{T_1}\right)\left(\frac{u}{U_1}\right)^2$$

Here $T_r=T_1(1+[(\gamma-1)/2]rM_1^2)$, with $r=0.89$ for solid surfaces. There is very little experimental data for the value of the recovery factor r for porous surfaces, but based on these limited data the formula $r=0.89(1+1.66F/c_f)^{-0.04}$ was used in the present calculations. All the measured temperatures were compared with the Crocco relation and with the temperatures predicted by the eddy viscosity model which does solve the energy equation. Except for the adverse pressure gradient with high injection rates the experimental results were in fair agreement with the predictions of the eddy viscosity model and with the Crocco relation. In general the predictions of the eddy viscosity model and the Crocco relation differ by less than 3° (i.e., by less than 1% of stagnation temperature) throughout the layer. Since the estimated error of the measured temperature can be up to 3°, it is impossible to differentiate between the two prediction methods.

Conclusions

From the preceding discussion it is clear that both prediction methods are satisfactory for the prediction of flows with combined injection and favorable pressure gradient, and for the initial development of the flow over a blown surface. However, neither method is completely satisfactory for flow with combined injection and adverse pressure gradient, or for the re-establishment of the flow downstream of a region of injection.

It would appear that for all the flows considered here the use of the Crocco relation is just as accurate as, and far more convenient than, the solution of the energy equation.

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Transition in a Streamwise Corner

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Nomenclature

U = streamwise mean velocity
X = streamwise coordinate along the corner
H = velocity profile shape parameter

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